

# Deformation of $\ell$ -adic Sheaves with Undeformed Local Monodromy \*

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## Abstract

Let  $X$  be a smooth connected algebraic curve over an algebraically closed field  $k$ . We study the deformation of  $\ell$ -adic Galois representations of the function field of  $X$  while keeping the local Galois representations at all places undeformed.

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## Introduction

In this paper, we work over an algebraically closed field  $k$  of characteristic  $p$  even though our results can be extended to non-algebraically closed fields. Let  $X$  be a smooth connected projective curve over  $k$ , let  $S$  be a finite closed subset of  $X$ , and let  $\ell$  be a prime number distinct from  $p$ . For any  $s \in S$ , let  $\eta_s$  be the generic point of the strict henselization of  $X$  at  $s$ . A lisse  $\overline{\mathbb{Q}}_\ell$ -sheaf  $\mathcal{F}$  on  $X - S$  is called *physically rigid* if for any lisse  $\overline{\mathbb{Q}}_\ell$ -sheaf  $\mathcal{G}$  on  $X - S$  with the property  $\mathcal{F}|_{\eta_s} \cong \mathcal{G}|_{\eta_s}$  for any closed point  $s$  in  $S$ , we have  $\mathcal{F} \cong \mathcal{G}$ . The lisse  $\overline{\mathbb{Q}}_\ell$ -sheaf  $\mathcal{F}$  on  $X - S$  corresponds to a Galois representation

$$\rho : \text{Gal}(\overline{K(X)}/K(X)) \rightarrow \text{GL}(n, \overline{\mathbb{Q}}_\ell)$$

of the function field  $K(X)$  unramified everywhere on  $X - S$ .  $\mathcal{F}$  is physically rigid if and only if for any Galois representation  $\rho'$  of  $\text{Gal}(\overline{K(X)}/K(X))$  such that  $\rho'$  and  $\rho$  induce isomorphic Galois representations of the local field obtained by taking completion of  $K(X)$  at any place of  $K(X)$ , we have  $\rho \cong \rho'$ . In another words, a physically rigid sheaf  $\mathcal{F}$  is completely determined by all the Galois representations of local fields defined by  $\mathcal{F}$ . To get a good notion of rigidity, we have to assume  $X = \mathbb{P}_k^1$ . Indeed, if  $X$  has genus  $g \geq 1$ , then there exists a lisse  $\overline{\mathbb{Q}}_\ell$ -sheaf  $\mathcal{L}$  of rank 1 on  $X$  such that  $\mathcal{L}^{\otimes n}$  is nontrivial for all  $n$ . For any lisse  $\overline{\mathbb{Q}}_\ell$ -sheaf  $\mathcal{F}$  on  $X - S$ , the lisse sheaf  $\mathcal{G} = \mathcal{F} \otimes \mathcal{L}$  is not

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isomorphic to  $\mathcal{F}$  since they have non-isomorphic determinant, but  $\mathcal{F}|_{\eta_s} \cong \mathcal{G}|_{\eta_s}$  for all  $s \in X$ . Hence  $\mathcal{F}$  is not rigid. A lisse  $\overline{\mathbb{Q}_\ell}$ -sheaf  $\mathcal{F}$  on  $X - S$  is called *cohomologically rigid* if we have

$$H^1(X, j_* \mathcal{E}nd(\mathcal{F})) = 0,$$

where  $j : X - S \hookrightarrow X$  is the canonical open immersion. In [7, 5.0.2], Katz shows that for an irreducible lisse sheaf, cohomological rigidity implies physical rigidity. It is conjectured that the converse is true.

In [2, Theorem 4.10], Bloch and Esnault study deformations of locally free  $\mathcal{O}_{X-S}$ -modules provided with connections while keeping local (formal) data undeformed, and they prove that physical rigidity and cohomological rigidity are equivalent for locally free  $\mathcal{O}_{X-S}$ -modules provided with connections. Motivated by their results, in this paper, we study the deformation of lisse  $\ell$ -adic sheaves while keeping the local monodromy undeformed. More precisely, Let  $F$  be any one of the following fields: a finite extension of the finite field  $\mathbb{F}_\ell$  with  $\ell$  elements, an algebraic closure  $\overline{\mathbb{F}_\ell}$  of  $\mathbb{F}_\ell$ , a finite extension of the  $\ell$ -adic number field  $\mathbb{Q}_\ell$ , or an algebraic closure  $\overline{\mathbb{Q}_\ell}$  of  $\mathbb{Q}_\ell$ . Let  $\mathcal{F}$  be a lisse  $F$ -sheaf on  $X - S$ . In this paper, we study the deformation of  $\mathcal{F}$  so that  $\mathcal{F}|_{\eta_s}$  ( $s \in S$ ) remain undeformed.

Let  $\eta$  be a generic point of  $X$ . We define an  $F$ -representation of  $\pi_1(X - S, \bar{\eta})$  of rank  $r$  to be a homomorphism  $\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(F^r)$  such that the following conditions holds: If  $F$  is a finite extension of  $\mathbb{F}_\ell$  or  $\mathbb{Q}_\ell$ , we require  $\rho$  to be continuous, where the topology on  $\mathrm{GL}(F^r)$  is the discrete topology if  $F$  is a finite field, and is induced by the  $\ell$ -adic topology on  $F$  if  $F$  is a finite extension of  $\mathbb{Q}_\ell$ ; if  $F$  an algebraic closure of  $\mathbb{F}_\ell$  (resp.  $\mathbb{Q}_\ell$ ), we require the existence of a finite extension  $E$  of  $\mathbb{F}_\ell$  (resp.  $\mathbb{Q}_\ell$ ) such that  $\rho$  factors through a continuous homomorphism  $\pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(E^r)$ . Let  $V = \mathcal{F}_{\bar{\eta}}$ . Then the lisse  $F$ -sheaf  $\mathcal{F}$  on  $X - S$  defines an  $F$ -representation

$$\rho_0 : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(V).$$

Fix an embedding  $\mathrm{Gal}(\bar{\eta}_s/\eta_s) \hookrightarrow \pi_1(X - S, \bar{\eta})$  for each  $s \in S$ . Our problem can be interpreted as the deformation of the representation  $\rho_0$  so that  $\rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$  ( $s \in S$ ) remain undeformed. Our treatment is similar to Mazur's theory of deformation of Galois representations ([8]).

Denote by  $\mathcal{C}$  the category of Artinian local  $F$ -algebras with residue field  $F$ . Morphisms in  $\mathcal{C}$  are  $F$ -algebra homomorphisms. Using the fact that the maximal ideal of an Artinian local ring coincides with its nilpotent radical, one can check that morphisms in  $\mathcal{C}$  are necessarily local homomorphisms and they induce the identity  $\mathrm{id}_F$  on the residue field. If  $A$  is an object in  $\mathcal{C}$ , we denote by  $\mathfrak{m}_A$  the maximal ideal of  $A$ . A homomorphism  $\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A^r)$  is called a representation if by regarding  $A^r$  as a finite dimensional  $F$ -vector space,  $\rho$  is an  $F$ -representation of  $\pi_1(X - S, \bar{\eta})$ .

Let  $\rho_1, \rho_2 : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A^r)$  be two representations. For any subgroup  $G$  of  $\pi_1(X - S, \bar{\eta})$ , we say  $\rho_1|_G$  and  $\rho_2|_G$  are *equivalent* if there exists  $P \in \mathrm{GL}(A^r)$  such that  $P^{-1}\rho_1(g)P = \rho_2(g)$  for all  $g \in G$ . We say  $\rho_1|_G$  and  $\rho_2|_G$  are *strictly equivalent* if the above condition holds for some  $P$  with the property  $P \equiv I \pmod{\mathfrak{m}_A}$ . We write  $\rho_1|_G \cong \rho_2|_G$  if they are equivalent.

Fix an  $F$ -representation  $\rho_0 : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(F^r)$ . For any  $A \in \mathrm{ob}\mathcal{C}$ , define  $R(A)$  to be the set of strict equivalent classes of representations  $\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A^r)$  such that  $\rho \equiv \rho_0 \pmod{\mathfrak{m}_A}$  and  $\rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} \cong \rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$  for all  $s \in S$ . Each element in  $R(A)$  is called a *deformation* of  $\rho_0$  with  $\rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$  being undeformed. The main result of this paper is the following theorem.

**Theorem 0.1.** *Assume all elements in the set  $\mathrm{End}_{F[\pi_1(X-S, \bar{\eta})]}(F^r)$  are scalar multiplications, where  $F^r$  is considered as an  $F[\pi_1(X - S, \bar{\eta})]$ -module through the representation  $\rho_0$ . (This condition holds if  $\rho_0$  is absolutely irreducible by Schur's lemma).*

(i) *The functor  $R : \mathcal{C} \rightarrow (\mathrm{Sets})$  is pro-representable, that is, there exist a complete noetherian local  $F$ -algebra  $R_{\mathrm{univ}}$  with residue field  $F$  and a homomorphism*

$$\rho_{\mathrm{univ}} : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(R_{\mathrm{univ}}^r)$$

*with the properties  $\rho_{\mathrm{univ}} \equiv \rho_0 \pmod{\mathfrak{m}_{R_{\mathrm{univ}}}}$  and  $\rho_{\mathrm{univ}}|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} \cong \rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$  for all  $s \in S$ , such that the homomorphism  $\pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}((R_{\mathrm{univ}}/\mathfrak{m}_{R_{\mathrm{univ}}}^m)^r)$  induced by  $\rho_{\mathrm{univ}}$  are representations for all positive integers  $m$ , and for any element  $\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A^r)$  in  $R(A)$ , there exists a unique local  $F$ -homomorphism  $R_{\mathrm{univ}} \rightarrow A$  which brings  $\rho_{\mathrm{univ}}$  to  $\rho$ . We call  $R_{\mathrm{univ}}$  the **universal deformation ring** of  $\rho_0$  and  $\rho_{\mathrm{univ}}$  the **universal deformation**.*

(ii) *Let  $F[\epsilon]$  be the ring of dual numbers over  $F$ . The tangent space  $R(F[\epsilon])$  of the functor  $R$  is isomorphic to  $H^1(X, j_*\mathcal{E}nd(\mathcal{F}))$ , where  $j : X - S \hookrightarrow X$  is the open immersion, and  $\mathcal{F}$  is the lisse  $F$ -sheaf on  $X$  corresponding to the representation  $\rho_0$ .*

(iii)  *$R_{\mathrm{univ}}$  is isomorphic to the formal power series ring  $F[[t_1, \dots, t_m]]$  with  $m = \dim_F H^1(X, j_*\mathcal{E}nd(\mathcal{F}))$ .*

(iv) *If we don't assume elements in  $\mathrm{End}_{F[\pi_1(X-S, \bar{\eta})]}(F^r)$  are scalar multiplications, then  $F$  has a pro-representable hull, that is, there exist a complete noetherian local  $F$ -algebra  $R_{\mathrm{univ}}$  with residue field  $F$  and a homomorphism  $\rho_{\mathrm{univ}} : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(R_{\mathrm{univ}}^r)$  with the properties  $\rho_{\mathrm{univ}} \equiv \rho_0 \pmod{\mathfrak{m}_{R_{\mathrm{univ}}}}$  and  $\rho_{\mathrm{univ}}|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} \cong \rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$  for all  $s \in S$ , such that for any element  $\rho$  in  $R(A)$ , there exists a (not necessarily unique) local  $F$ -homomorphism  $R_{\mathrm{univ}} \rightarrow A$  which brings  $\rho_{\mathrm{univ}}$  to  $\rho$ .*

We will prove (i), (ii) and (iv) in §1, and prove (iii) in §2.

# 1 Proof of the main theorem

In this section, we prove study the pro-representability of the functor  $R$  using Schlessinger's criterion ([9]). We start with a series of lemmas.

**Lemma 1.1.** *Let  $G$  be a group, let  $F$  be a field, and let  $V_1$  and  $V_2$  be  $F[G]$ -modules which are finite dimensional over  $F$ . Then for any commutative  $F$ -algebra  $A$ , we have a canonical isomorphism*

$$\mathrm{Hom}_{F[G]}(V_1, V_2) \otimes_F A \cong \mathrm{Hom}_{A[G]}(V_1 \otimes_F A, V_2 \otimes_F A).$$

*Proof.* Fix bases for the  $F$ -vector spaces  $V_1$  and  $V_2$ . They give rise to bases for the  $A$ -modules  $V_1 \otimes_F A$  and  $V_2 \otimes_F A$ . Write homomorphisms between these  $A$ -modules in terms of matrices using these bases. Suppose  $T : V_1 \otimes_F A \rightarrow V_2 \otimes_F A$  is an  $A[G]$ -module homomorphism. Then we can write

$$T = \sum_i a_i E_i$$

such that  $a_i \in A$  are linearly independent over  $F$  and  $E_i$  are matrices with entries in  $F$ . For any  $g \in G$ , let  $M_g$  and  $N_g$  be the matrices of the action of  $g$  on  $V_1$  and  $V_2$ , respectively. Note that entries of  $M_g$  and  $N_g$  lie in  $F$ . We have

$$\sum_i a_i (E_i M_g - N_g E_i) = 0.$$

By the linear independence of  $a_i$  over  $F$ , we have  $E_i M_g = N_g E_i$  for all  $i$  and all  $g \in G$ . So  $E_i$  define  $F[G]$ -module homomorphisms from  $V_1$  and  $V_2$ . This shows that the canonical map

$$\mathrm{Hom}_{F[G]}(V_1, V_2) \otimes_F A \rightarrow \mathrm{Hom}_{A[G]}(V_1 \otimes_F A, V_2 \otimes_F A)$$

is surjective. The injectivity of this map follows from the fact that  $\mathrm{Hom}_{F[G]}(V_1, V_2) \otimes_F A$  (resp.  $\mathrm{Hom}_{A[G]}(V_1 \otimes_F A, V_2 \otimes_F A)$ ) can be considered as a subspace of  $\mathrm{Hom}_F(V_1, V_2) \otimes_F A$  (resp.  $\mathrm{Hom}_A(V_1 \otimes_F A, V_2 \otimes_F A)$ ), and that

$$\mathrm{Hom}_F(V_1, V_2) \otimes_F A \cong \mathrm{Hom}_A(V_1 \otimes_F A, V_2 \otimes_F A).$$

(The last isomorphism follows from the fact  $V_1 \cong F^r$  ( $r = \dim_F V$ ).) □

**Lemma 1.2.** *Let  $A$  be an Artinian local ring,  $F = A/\mathfrak{m}_A$  the residue field,  $\rho : G \rightarrow \mathrm{GL}(A^r)$  a homomorphism, and  $\rho_0 : G \rightarrow \mathrm{GL}(F^r)$  the homomorphism defined by  $\rho$  modulo  $\mathfrak{m}_A$ . Regard  $A^r$  (resp.  $F^r$ ) as a module over  $A[G]$  (resp.  $F[G]$ ) through the representation  $\rho$  (resp.  $\rho_0$ .) Suppose all elements in  $\mathrm{End}_{F[G]}(F^r)$  are scalar multiplications. Then all elements in  $\mathrm{End}_{A[G]}(A^r)$  are scalar multiplications.*

*Proof.* Recall that an epimorphism  $A \rightarrow B$  of Artinian local rings is called *small* if its kernel  $\mathfrak{a}$  is a principal ideal with the property  $\mathfrak{a}\mathfrak{m}_A = 0$ . Any epimorphism of Artinian local rings can be written as a composite of a series of small epimorphisms. To prove the lemma, it suffices to prove the following statement: If  $\phi : A \rightarrow B$  is a small extension, and all elements in  $\text{End}_{B[G]}(B^r)$  and all elements in  $\text{End}_{F[G]}(F^r)$  are scalar multiplications, then all elements in  $\text{End}_{A[G]}(A^r)$  are scalar multiplications. When  $\phi$  is an isomorphism, this is obvious. Suppose  $\phi$  is not an isomorphism. Let  $t$  be a generator of the kernel of  $\phi$ . Since  $t\mathfrak{m}_A = 0$ , multiplication by  $t$  induces a homomorphism

$$A/\mathfrak{m}_A \rightarrow tA$$

which is necessarily injective. So if  $ta_1 = ta_2$ , then  $a_1 \equiv a_2 \pmod{\mathfrak{m}_A}$ . Let  $P \in \text{End}_{A[G]}(A^r)$ . Then  $\phi(P)$  lies in  $\text{End}_{B[G]}(B^r)$ . By our assumption,  $\phi(P)$  is scalar. So there exist  $a \in A$  and a matrix  $\Delta$  with entries in  $A$  such that

$$P = aI + t\Delta.$$

For any  $g \in G$ , from the fact that  $\rho(g)P = P\rho(g)$ , we get  $t\rho(g)\Delta = t\Delta\rho(g)$ . By our previous discussion, this implies that  $\rho(g)\Delta \equiv \Delta\rho(g) \pmod{\mathfrak{m}_A}$ . Hence modulo  $\mathfrak{m}_A$ ,  $\Delta$  defines an element in  $\text{End}_{F[G]}(F^r)$ . By our assumption, there exist  $a' \in A$  and a matrix  $\Delta'$  with entries in  $\mathfrak{m}_A$  such that

$$\Delta = a'I + \Delta'.$$

As  $t\mathfrak{m}_A = 0$ , we have

$$P = aI + t\Delta = aI + t(a'I + \Delta') = (a + ta')I.$$

Hence  $P$  is scalar. □

From now on let  $F$  be a finite extension of the finite field  $\mathbb{F}_\ell$  or  $\mathbb{Q}_\ell$ , or an algebraic closure of such a field, and let  $\mathcal{C}$  be the category of Artinian local  $F$ -algebras with residue field  $F$ . Let  $X$  be a connected smooth projective curve over an algebraically closed field  $k$  of characteristic  $p$  distinct from  $\ell$ ,  $S$  a finite closed subset of  $X$ ,  $\eta$  the generic point of  $X$ , and  $\rho_0 : \pi_1(X - S, \bar{\eta}) \rightarrow \text{GL}(F^r)$  an  $F$ -representation of  $\pi_1(X - S, \bar{\eta})$ . Define the functor  $R : \mathcal{C} \rightarrow (\text{Sets})$  as in Theorem 0.1. We will apply [9, Theorem 2.11] to this functor. Let  $\phi' : A' \rightarrow A$  and  $\phi'' : A'' \rightarrow A$  be morphisms in  $\mathcal{C}$ , and consider the map

$$(1) \quad R(A' \times_A A'') \rightarrow R(A') \times_{R(A)} R(A'').$$

**Lemma 1.3.** *Suppose  $\phi'' : A'' \rightarrow A$  is surjective. Then the map (1) is surjective.*

*Proof.* Let  $\rho' : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A'^r)$  and  $\rho'' : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A''^r)$  be elements in  $R(A')$  and  $R(A'')$ , respectively, such that they have the same image in  $R(A)$ . Then there exists  $P \in \mathrm{GL}(A^r)$  such that  $P \equiv I \pmod{\mathfrak{m}_A}$  and

$$\phi' \circ \rho' = P^{-1}(\phi'' \circ \rho'')P.$$

Here for convenience, we denote the homomorphism  $\mathrm{GL}(A'^r) \rightarrow \mathrm{GL}(A^r)$  (resp.  $\mathrm{GL}(A''^r) \rightarrow \mathrm{GL}(A^r)$ ) induced by  $\phi'$  (resp.  $\phi''$ ) also by  $\phi'$  (resp.  $\phi''$ ). Since  $\phi''$  is surjective. There exists  $P'' \in \mathrm{GL}(A''^r)$  such that  $\phi''(P'') = P$  and  $P'' \equiv I \pmod{\mathfrak{m}_{A''}}$ . Replacing  $\rho''$  by  $P''^{-1}\rho''P''$ , we may assume

$$\phi' \circ \rho' = \phi'' \circ \rho''.$$

We can then define a representation

$$\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}\left((A' \times_A A'')^r\right)$$

such that  $\rho$  is mapped to  $\rho'$  and  $\rho''$  under the two projections  $A' \times_A A'' \rightarrow A'$  and  $A' \times_A A'' \rightarrow A''$ , respectively. It is clear that  $\rho \equiv \rho_0 \pmod{\mathfrak{m}_{A' \times_A A''}}$ . To prove the lemma, it remains to verify that

$$\rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} \cong \rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$$

for any  $s \in S$ .

There exist  $P'_s \in \mathrm{GL}(A'^r)$  and  $P''_s \in \mathrm{GL}(A''^r)$  such that

$$\rho'|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} = P'^{-1}_s(\rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)})P'_s, \quad \rho''|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} = P''^{-1}_s(\rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)})P''_s.$$

Since  $\phi' \circ \rho' = \phi'' \circ \rho''$ , we have

$$\phi'(P'_s)^{-1}(\rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)})\phi'(P'_s) = \phi''(P''_s)^{-1}(\rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)})\phi''(P''_s).$$

So  $\phi'(P'_s)\phi''(P''_s)^{-1} \in \mathrm{GL}(A^r)$  defines an automorphism of the representation

$$\mathrm{Gal}(\bar{\eta}_s/\eta_s) \xrightarrow{\rho_0} \mathrm{GL}(F^r) \hookrightarrow \mathrm{GL}(A^r)$$

obtained from  $\rho_0$  by scalar extension from  $F$  to  $A$ . By Lemma 1.1, we have

$$\begin{aligned} \mathrm{End}_{A[\mathrm{Gal}(\bar{\eta}_s/\eta_s)]}(A^r) &\cong \mathrm{End}_{F[\mathrm{Gal}(\bar{\eta}_s/\eta_s)]}(F^r) \otimes_F A, \\ \mathrm{End}_{A''[\mathrm{Gal}(\bar{\eta}_s/\eta_s)]}(A''^r) &\cong \mathrm{End}_{F[\mathrm{Gal}(\bar{\eta}_s/\eta_s)]}(F^r) \otimes_F A'', \end{aligned}$$

where  $A^r$  (resp.  $A''^r$ ) is considered as an  $A[\text{Gal}(\bar{\eta}_s/\eta_s)]$ -module (resp.  $A''[\text{Gal}(\bar{\eta}_s/\eta_s)]$ -module) through the representation  $\rho_0|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$ . Since  $\phi'' : A'' \rightarrow A$  is surjective, the canonical homomorphism

$$\text{End}_{A''[\text{Gal}(\bar{\eta}_s/\eta_s)]}(A''^r) \rightarrow \text{End}_{A[\text{Gal}(\bar{\eta}_s/\eta_s)]}(A^r)$$

is surjective. Using the fact that an endomorphism  $Q$  in  $\text{End}_{A[\text{Gal}(\bar{\eta}_s/\eta_s)]}(A^r)$  is an isomorphism if and only if  $Q$  induces an isomorphism of  $(A/\mathfrak{m}_A)^r$ , we see that

$$\text{Aut}_{A''[\text{Gal}(\bar{\eta}_s/\eta_s)]}(A''^r) \rightarrow \text{Aut}_{A[\text{Gal}(\bar{\eta}_s/\eta_s)]}(A^r)$$

is surjective. We have shown that  $\phi'(P'_s)\phi''(P''_s)^{-1}$  lies in  $\text{Aut}_{A[\text{Gal}(\bar{\eta}_s/\eta_s)]}(A^r)$ . So there exists  $Q''_s \in \text{Aut}_{A''[\text{Gal}(\bar{\eta}_s/\eta_s)]}(A''^r)$  such that

$$\phi''(Q''_s) = \phi'(P'_s)\phi''(P''_s)^{-1}.$$

We then have

$$\phi'(P'_s) = \phi''(Q''_s P''_s),$$

$$\rho'|_{\text{Gal}(\bar{\eta}_s/\eta_s)} = P_s'^{-1}(\rho_0|_{\text{Gal}(\bar{\eta}_s/\eta_s)})P'_s, \quad \rho''|_{\text{Gal}(\bar{\eta}_s/\eta_s)} = (Q''_s P''_s)^{-1}(\rho_0|_{\text{Gal}(\bar{\eta}_s/\eta_s)})Q''_s P''_s.$$

We can find  $P_s \in \text{GL}\left((A' \times_A A'')^r\right)$  which is mapped to  $P'_s$  and  $Q''_s P''_s$  under the two projections  $A' \times_A A'' \rightarrow A'$  and  $A' \times_A A'' \rightarrow A''$ , respectively. We then have

$$\rho|_{\text{Gal}(\bar{\eta}_s/\eta_s)} = P_s^{-1}(\rho_0|_{\text{Gal}(\bar{\eta}_s/\eta_s)})P_s.$$

This finishes the proof of the lemma.  $\square$

**Lemma 1.4.** *Suppose  $\phi'' : A'' \rightarrow A$  is surjective. If one of the following conditions holds, then the map (1) is bijective.*

- (a) *All elements in  $\text{End}_{F[\pi_1(X-S, \bar{\eta})]}(F^r)$  are scalar multiplications.*
- (b)  *$A = F$  and  $A'' = F[\epsilon]$  is the ring of dual numbers over  $F$ .*

*Proof.* By Lemma 1.3, it suffices to show (1) is injective. Let  $\rho_1, \rho_2 : \pi_1(X-S, \bar{\eta}) \rightarrow \text{GL}_{A' \times_A A''}\left((A' \times_A A'')^r\right)$  be two elements in  $R(A' \times_A A'')$  such that they have same images in both  $R(A')$  and  $R(A'')$ . Let  $\psi' : A' \times_A A'' \rightarrow A'$  and  $\psi'' : A' \times_A A'' \rightarrow A''$  be the projections. Then there exist  $P' \in \text{GL}(A'^r)$  and  $P'' \in \text{GL}(A''^r)$  such that

$$\begin{aligned} P' &\equiv I \pmod{\mathfrak{m}_{A'}}, & P'' &\equiv I \pmod{\mathfrak{m}_{A''}}, \\ \psi'(\rho_1) &= P'^{-1}\psi'(\rho_2)P', & \psi''(\rho_1) &= P''^{-1}\psi''(\rho_2)P''. \end{aligned}$$

We then have

$$\phi' \psi'(\rho_1) = \phi'(P')^{-1} \phi' \psi'(\rho_2) \phi'(P'), \quad \phi'' \psi''(\rho_1) = \phi''(P'')^{-1} \phi'' \psi''(\rho_2) \phi''(P'').$$

We have

$$\phi' \psi'(\rho_1) = \phi'' \psi''(\rho_1), \quad \phi' \psi'(\rho_2) = \phi'' \psi''(\rho_2).$$

Set  $\rho = \phi' \psi'(\rho_2) = \phi'' \psi''(\rho_2)$ . Then we have

$$\left( \phi'(P') \phi''(P'')^{-1} \right)^{-1} \rho \left( \phi'(P') \phi''(P'')^{-1} \right) = \rho.$$

First we work under the condition (a). By Lemma 1.2,  $\phi'(P') \phi''(P'')^{-1}$  must be a scalar matrix. Choose a scalar matrix  $a'' I$  such that

$$\phi'(P') \phi''(P'')^{-1} = \phi''(a'') I,$$

where  $a''$  is a unit in  $A''$  and  $a'' \equiv 1 \pmod{\mathfrak{m}_{A''}}$ . We have  $\phi'(P') = \phi''(a'' P'')$ . So we can find  $Q \in \text{GL}\left((A' \times_A A'')^r\right)$  such that

$$\psi'(Q) = P', \quad \psi''(Q) = a'' P'', \quad Q \equiv I \pmod{\mathfrak{m}_{A' \times_A A''}}.$$

As

$$\psi'(\rho_1) = \psi'(Q)^{-1} \psi'(\rho_2) \psi'(Q), \quad \psi''(\rho_1) = \psi''(Q)^{-1} \psi''(\rho_2) \psi''(Q),$$

we have  $\rho_1 = Q^{-1} \rho_2 Q$ . So  $\rho_1$  and  $\rho_2$  are strictly equivalent and they give rise to the same element in  $R(A' \times_A A'')$ . Hence the map (1) is injective.

Next we work under the condition (b). Since

$$P' \equiv I \pmod{\mathfrak{m}_{A'}}, \quad P'' \equiv I \pmod{\mathfrak{m}_{A''}},$$

we have  $\phi'(P') = \phi''(P'') = I$ . The above argument still works by taking  $a'' = 1$ . □

Let  $\phi' : A' \rightarrow A$  be a homomorphism in the category  $\mathcal{C}$  with kernel  $\mathfrak{a}$ , and let  $G$  be a group. Two homomorphisms  $\rho_i : G \rightarrow \text{GL}(A'^r)$  ( $i = 1, 2$ ) are called *strictly equivalent relative to  $\phi'$*  if  $\rho_1 \equiv \rho_2 \pmod{\mathfrak{a}}$  and there exists  $P \in \text{GL}(A'^r)$  such that  $P \equiv I \pmod{\mathfrak{a}}$  and  $P^{-1} \rho_1 P = \rho_2$ .

**Lemma 1.5.** *Let  $\phi' : A' \rightarrow A$  be a homomorphism in  $\mathcal{C}$  with kernel  $\mathfrak{a}$ .*

(i) *Let  $\rho_1, \rho_2 : \pi_1(X - S, \bar{\eta}) \rightarrow \text{GL}(A'^r)$  be two representation such that  $\rho_1 \equiv \rho_2 \pmod{\mathfrak{a}}$  and such that  $\rho_i \equiv \rho_0 \pmod{\mathfrak{m}_{A'}} (i = 1, 2)$ . Suppose all elements in  $\text{End}_{F[\pi_1(X - S, \bar{\eta})]}(F^r)$  are scalar*



multiplications. Then  $\rho_1$  is equivalent to  $\rho_2$  if and only if  $\rho_1$  is strictly equivalent to  $\rho_2$  relative to  $\phi'$ .

(ii) Suppose furthermore that  $\phi' : A' \rightarrow A$  is surjective. Let  $\rho_1 : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A^r)$  be a representation such that  $\rho_1 \equiv \rho_0 \pmod{\mathfrak{a}}$ . Then for any subgroup  $G$  of  $\pi_1(X - S, \bar{\eta})$ , we have that  $\rho_1|_G$  is equivalent to  $\rho_0|_G$  if and only if  $\rho_1|_G$  is strictly equivalent to  $\rho_0|_G$  relative to  $\phi'$ .

*Proof.*

(i) If  $\rho_1$  is equivalent to  $\rho_2$ , then we can find  $P \in \mathrm{GL}(A^r)$  such that  $P^{-1}\rho_1P = \rho_2$ . Modulo  $\mathfrak{a}$ , this equation implies that  $P_0^{-1}\rho P_0 = \rho$ , where  $P_0 \in \mathrm{GL}(A^r)$  is the image of  $P$  under the homomorphism  $\mathrm{GL}(A^r) \rightarrow \mathrm{GL}(A^r)$ , and  $\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A^r)$  is the representation such that  $\rho = \rho_1 \equiv \rho_2 \pmod{\mathfrak{a}}$ . If all elements in  $\mathrm{End}_{F[\pi_1(X-S, \bar{\eta})]}(F^r)$  are scalar multiplications, then by Lemma 1.2,  $P_0$  must be a scalar matrix. Let  $P' \in \mathrm{GL}(A^r)$  be a scalar matrix lifting  $P_0$ . Then we have  $PP'^{-1} \equiv I \pmod{\mathfrak{a}}$ , and  $(PP'^{-1})^{-1}\rho_1(PP'^{-1}) = \rho_2$ . So  $\rho_1$  is strictly equivalent to  $\rho_2$  relative to  $\phi'$ .

(ii) If  $\rho_1|_G$  is equivalent to  $\rho_0|_G$ , then we can find  $P \in \mathrm{GL}(A^r)$  such that  $P^{-1}\rho_1(g)P = \rho_0(g)$  for all  $g \in G$ . Modulo  $\mathfrak{a}$ , this equation implies that  $P_0^{-1}\rho_0(g)P_0 = \rho_0(g)$  for all  $g \in G$ , where  $P_0 \in \mathrm{GL}(A^r)$  is the image of  $P$  under the homomorphism  $\mathrm{GL}(A^r) \rightarrow \mathrm{GL}(A^r)$ . So we have  $P_0 \in \mathrm{Aut}_{A[G]}(A^r)$ , where  $A^r$  is regarded as a  $G$ -module through the representation  $\rho_0$ . As in the proof of Lemma 1.3, it follows from Lemma 1.1 that the canonical map

$$\mathrm{Aut}_{A[G]}(A^r) \rightarrow \mathrm{Aut}_{A[G]}(A^r)$$

is surjective if  $A' \rightarrow A$  is surjective. Here  $A^r$  is regarded as a  $G$ -module through the representation  $\rho_0$ . So we can find  $P' \in \mathrm{GL}(A^r)$  such that  $P' \equiv P_0 \pmod{\mathfrak{a}}$  and  $P'^{-1}\rho_0(g)P' = \rho_0(g)$  for all  $g \in G$ . We have  $PP'^{-1} \equiv I \pmod{\mathfrak{a}}$ , and  $(PP'^{-1})^{-1}\rho_1(g)(PP'^{-1}) = \rho_0(g)$  for all  $g \in G$ . So  $\rho_1|_G$  is strictly equivalent to  $\rho_0|_G$  relative to  $\phi'$ .  $\square$

Given an  $F$ -representation  $\pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(F^r)$ , we can talk about the cohomology groups  $H^i(\pi_1(X - S, \bar{\eta}), F^r)$ . Indeed, for any topological  $\pi_1(X - S, \bar{\eta})$ -modules  $M$ , we can define a chain complex  $C^*(\pi_1(X - S, \bar{\eta}), M)$  as in [10, I 2.2] by requiring  $C^i(\pi_1(X - S, \bar{\eta}), M)$  to be the group of continuous maps  $\pi_1(X - S, \bar{\eta})^i \rightarrow M$ , and we define

$$H^i(\pi_1(X - S, \bar{\eta}), M) \cong H^i(C^*(\pi_1(X - S, \bar{\eta}), M)).$$

This allows us to define  $H^i(\pi_1(X - S, \bar{\eta}), F^r)$  in the case where  $F$  is a finite extension of  $\mathbb{F}_\ell$  or  $\mathbb{Q}_\ell$ . In the case where  $F$  is a finite field,  $F^r$  is a finite discrete  $\pi_1(X - S, \bar{\eta})$ -module. The cohomology groups

of discrete  $\pi_1(X - S, \bar{\eta})$ -modules are studied in detail in [10]. In the case where  $F$  is a finite extension of  $\mathbb{Q}_\ell$ , let  $\Lambda$  be the integral closure of  $\mathbb{Z}_\ell$  in  $F$ , let  $\lambda$  be a uniformizer of  $\Lambda$ , and let  $L$  be a lattice in  $F^r$  which is stable under the action of  $\pi_1(X - S, \bar{\eta})$ . Then we have

$$\begin{aligned} C^*(\pi_1(X - S, \bar{\eta}), F^r) &\cong C^*(\pi_1(X - S, \bar{\eta}), L) \otimes_\Lambda F, \\ H^i(\pi_1(X - S, \bar{\eta}), F^r) &\cong H^i(\pi_1(X - S, \bar{\eta}), L) \otimes_\Lambda F. \end{aligned}$$

Moreover, we have

$$C^*(\pi_1(X - S, \bar{\eta}), L) \cong \varprojlim_n C^*(\pi_1(X - S, \bar{\eta}), L/\lambda^n L).$$

Using [5, 0<sub>III</sub> 13.2.3], one can show

$$H^i(\pi_1(X - S, \bar{\eta}), L) \cong \varprojlim_n H^i(\pi_1(X - S, \bar{\eta}), L/\lambda^n L).$$

Note that  $L/\lambda^n L$  are finite discrete  $\pi_1(X - S, \bar{\eta})$ -modules. Finally in the case where  $F$  is an algebraic closure of  $\mathbb{F}_\ell$  (resp.  $\mathbb{Q}_\ell$ ), choose a finite extension  $E$  of  $\mathbb{F}_\ell$  (resp.  $\mathbb{Q}_\ell$ ) contained in  $F$  such that the representation  $\pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(F^r)$  is obtained from a representation  $\pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(E^r)$  by scalar extension. We define

$$H^i(\pi_1(X - S, \bar{\eta}), F^r) = H^i(\pi_1(X - S, \bar{\eta}), E^r) \otimes_E F.$$

Note that for any finite extension  $E'$  of  $E$  contained in  $F$ , we have

$$\begin{aligned} C^i(\pi_1(X - S, \bar{\eta}), E'^r) &\cong C^i(\pi_1(X - S, \bar{\eta}), E^r) \otimes_E E', \\ H^i(\pi_1(X - S, \bar{\eta}), E'^r) &\cong H^i(\pi_1(X - S, \bar{\eta}), E^r) \otimes_E E', \end{aligned}$$

and  $H^i(\pi_1(X - S, \bar{\eta}), F^r)$  is isomorphic to the  $i$ -th cohomology group of the chain complex  $C'^*(\pi_1(X - S, \bar{\eta}), F^r)$  defined by

$$C'^*(\pi_1(X - S, \bar{\eta}), F^r) = \varinjlim_{E'} C^*(\pi_1(X - S, \bar{\eta}), E'^r),$$

where  $E'$  goes over the set of finite extensions of  $E$  contained in  $F$ . Note that for this chain complex,  $C'^i(\pi_1(X - S, \bar{\eta}), F^r)$  is contained in the group of all continuous maps  $\pi_1(X - S, \bar{\eta})^i \rightarrow F^r$ .

If  $M$  is a finite discrete  $\pi_1(X - S, \bar{\eta})$ -module, then  $M$  defines a locally constant etale sheaf  $\mathcal{M}$  on  $X - S$ . If  $M$  is a torsion discrete  $\pi_1(X - S, \bar{\eta})$ -module, then  $M$  is a direct limit of finite discrete  $\pi_1(X - S, \bar{\eta})$ -modules, and hence  $M$  also defines an etale sheaf  $\mathcal{M}$  on  $X - S$ . By [6, XI 5], we have

$$H^1(\pi_1(X - S, \bar{\eta}), M) \cong H^1(X - S, \mathcal{M})$$

for any finite discrete  $\pi_1(X - S, \bar{\eta})$ -module  $M$ , and hence for any torsion discrete  $\pi_1(X - S, \bar{\eta})$ -module  $M$  by [1, VII 5.7] and [10, Proposition I 8]. It follows from the above the discussion that we have the following:

**Lemma 1.6.** *Given an  $F$ -representation  $\pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(F^r)$ , let  $\mathcal{F}$  be the corresponding  $F$ -sheaf on  $X - S$ . We have*

$$H^1(\pi_1(X - S, \bar{\eta}), F^r) \cong H^1(X - S, \mathcal{F}).$$

**Lemma 1.7.** *Let  $\phi' : A' \rightarrow A$  be a nonzero homomorphism in the category  $\mathcal{C}$  such that its kernel  $\mathfrak{a}$  has the property  $\mathfrak{m}_{A'}\mathfrak{a} = 0$ . Then  $\mathfrak{a}$  can be regarded as a vector space over  $F$  and  $\mathfrak{a}^2 = 0$ . Let  $\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A^r)$  be a representation such that  $\rho \equiv \rho_0 \pmod{\mathfrak{m}_A}$  and such that  $\rho$  can be lifted to a representation  $\pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A'^r)$ . Then the set of strictly equivalent classes relative to  $\phi'$  of such representations lifting  $\rho$  can be identified with the set  $H^1(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}(\rho_0)) \otimes_F \mathfrak{a}$ , where  $\mathrm{Ad}(\rho_0)$  is the  $F$ -vector space of  $r \times r$  matrices with entries in  $F$  on which  $\pi_1(X - S, \bar{\eta})$  acts by the composition of  $\rho_0$  with the adjoint representation of  $\mathrm{GL}(F^r)$ .*

*Proof.* Fix a representation  $\rho_1 : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A'^r)$  lifting  $\rho$ . Any representation  $\rho_2$  lifting  $\rho$  can be written in the form

$$\rho_2(g) = \rho_1(g) + \delta(g)\rho_1(g),$$

where  $\delta(g)$  ( $g \in \pi_1(X - S, \bar{\eta})$ ) are  $r \times r$  matrices with entries in  $\mathfrak{a}$ , and they define a continuous map  $\delta : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{Ad}(\rho_0) \otimes_F \mathfrak{a}$ . Using the fact that  $\rho_2(g_1g_2) = \rho_2(g_1)\rho_2(g_2)$ , one can verify  $\delta$  is a 1-cocycle. Conversely, for any 1-cocycle  $\delta : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{Ad}(\rho_0) \otimes_F \mathfrak{a}$ , the map  $\rho_1 + \delta\rho_1 : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A'^r)$  is a representation lifting  $\rho$ . Suppose  $\delta, \delta' : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{Ad}(\rho_0) \otimes_F \mathfrak{a}$  are two 1-cocycles such that  $\delta - \delta'$  differs by a 1-coboundary, that is,

$$\delta'(g) - \delta(g) = \rho_1(g)M\rho_1(g)^{-1} - M$$

for some  $r \times r$  matrix  $M$  with entries in  $\mathfrak{a}$ . Let  $\rho_2 = \rho_1 + \delta\rho_1$  and  $\rho'_2 = \rho_1 + \delta'\rho_1$ . Then we have

$$\rho'_2 = (I - M)\rho_2(I + M) = (I + M)^{-1}\rho'_2(I + M).$$

So  $\rho_2$  and  $\rho'_2$  are strictly equivalent relative to  $\phi'$ . Conversely, if  $\rho_2$  and  $\rho'_2$  are two representations lifting  $\rho$  which are strictly equivalent relative to  $\phi'$ , then the 1-cocycles  $\delta$  and  $\delta'$  defined by  $\rho_2 = \rho_1 + \delta\rho_1$  and  $\rho'_2 = \rho_1 + \delta'\rho_1$  differ by a 1-coboundary. This proves our assertion.  $\square$

**Lemma 1.8.** *Let  $j : X - S \hookrightarrow X$  be the canonical open immersion, and let  $\mathcal{G}$  be a lisse  $F$ -sheaf on  $X - S$ . Then we have a canonical exact sequence*

$$0 \rightarrow H^1(X, j_*\mathcal{G}) \rightarrow H^1(X - S, \mathcal{G}) \rightarrow \bigoplus_{s \in S} H^1(\eta_s, \mathcal{G}|_{\eta_s}) \rightarrow H^2(X, j_*\mathcal{G}).$$

*Proof.* Let  $\Delta$  be the mapping cone of the canonical morphism  $j_*\mathcal{G} \rightarrow Rj_*\mathcal{G}$ . We have a distinguished triangle

$$j_*\mathcal{G} \rightarrow Rj_*\mathcal{G} \rightarrow \Delta \rightarrow .$$

It gives rise to a long exact sequence

$$j_*\mathcal{G} \xrightarrow{\cong} j_*\mathcal{G} \rightarrow \mathcal{H}^0(\Delta) \rightarrow 0 \rightarrow R^1j_*\mathcal{G} \rightarrow \mathcal{H}^1(\Delta) \rightarrow 0.$$

It follows that  $\mathcal{H}^i(\Delta) = 0$  for  $i \neq 1$  and  $\mathcal{H}^1(\Delta) \cong R^1j_*\mathcal{G}$ . Note that  $R^1j_*\mathcal{G}$  is a punctured sheaf supported on  $S$ , and for any  $s \in S$ , we have

$$(R^1j_*\mathcal{G})_{\bar{s}} \cong H^1(\eta_s, \mathcal{G}|_{\eta_s}).$$

It follows that

$$H^0(X, \Delta) = 0, \quad H^1(X, \Delta) \cong \bigoplus_{s \in S} H^1(\eta_s, \mathcal{G}|_{\eta_s}).$$

Taking the long exact sequence of cohomology groups associated to the above distinguished triangle, we get a long exact sequence

$$\begin{array}{ccccccc} H^0(X, \Delta) & \rightarrow & H^1(X, j_*\mathcal{G}) & \rightarrow & H^1(X, Rj_*\mathcal{G}) & \rightarrow & H^1(X, \Delta) \rightarrow H^2(X, j_*\mathcal{G}). \\ \wr \parallel & & & & \wr \parallel & & \wr \parallel \\ 0 & & & & H^1(X - S, \mathcal{G}) & & \bigoplus_{s \in S} H^1(\eta_s, \mathcal{G}|_{\eta_s}) \end{array}$$

Our assertion follows.  $\square$

**Lemma 1.9.** *We have a canonical isomorphism  $R(F[\epsilon]) \cong H^1(X, j_*\mathcal{E}nd(\mathcal{F}))$ , where  $j : X - S \hookrightarrow X$  is the open immersion, and  $\mathcal{F}$  is the lisse  $F$ -sheaf on  $X$  corresponding to the representation  $\rho_0$ .*

*Proof.* By Lemma 1.7,  $H^1(\pi_1(X - S, \bar{\eta}), \text{Ad}(\rho_0))$  can be identified with the set of strict equivalent classes of representations  $\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \text{GL}((F[\epsilon])^r)$  with the property  $\rho \equiv \rho_0 \pmod{\epsilon}$ . Similarly,  $H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0))$  can be identified with the set of strict equivalent classes of representations  $\rho : \text{Gal}(\bar{\eta}_s/\eta_s) \rightarrow \text{GL}((F[\epsilon])^r)$  with the property  $\rho \equiv \rho_0 \pmod{\epsilon}$ .

Let's describe the kernel of the canonical homomorphism

$$H^1(\pi_1(X - S, \bar{\eta}), \text{Ad}(\rho_0)) \rightarrow H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0)).$$

Let  $\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}((F[\epsilon])^r)$  be a representation with the property  $\rho \equiv \rho_0 \pmod{\epsilon}$ . The 1-cocycle  $M : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{End}((F[\epsilon])^r)$  defined by  $\rho = \rho_0 + M\rho_0$  becomes a 1-coboundary with respect to the group  $\mathrm{Gal}(\bar{\eta}_s/\eta_s)$  if and only if  $\rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$  is strictly equivalent to  $\rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$ . By Lemma 1.5 (ii), this is equivalent to saying that  $\rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$  is equivalent to  $\rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$ . Therefore the kernel of the canonical homomorphism

$$H^1(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}(\rho_0)) \rightarrow H^1(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}(\rho_0))$$

can be identified with the set of strict equivalent classes of representations  $\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}((F[\epsilon])^r)$  such that  $\rho \equiv \rho_0 \pmod{\epsilon}$  and  $\rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$  is equivalent to  $\rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$ .

By Lemma 1.8,  $H^1(X, j_*\mathcal{E}nd(\mathcal{F}))$  can be identified with the kernel of the canonical homomorphism

$$H^1(X - S, \mathcal{E}nd(\mathcal{F})) \rightarrow \bigoplus_{s \in S} H^1(\eta_s, \mathcal{E}nd(\mathcal{F})|_{\eta_s}).$$

By Lemma 1.6, we have canonical isomorphisms

$$H^1(X - S, \mathcal{E}nd(\mathcal{F})) \cong H^1(\pi_1(X - S, \bar{\eta}), \mathrm{Ad}(\rho_0)), \quad H^1(\eta_s, \mathcal{E}nd(\mathcal{F})|_{\eta_s}) \cong H^1(\mathrm{Gal}(\bar{\eta}_s/\eta_s), \mathrm{Ad}(\rho_0)).$$

Combined with the above discussion, we find that  $H^1(X, j_*\mathcal{E}nd(\mathcal{F}))$  can be canonically identified with the set of strict equivalent classes of representations  $\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}((F[\epsilon])^r)$  such that  $\rho \equiv \rho_0 \pmod{\epsilon}$  and  $\rho|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)} \cong \rho_0|_{\mathrm{Gal}(\bar{\eta}_s/\eta_s)}$  for all  $s \in S$ . That is, we have  $H^1(X, j_*\mathcal{E}nd(\mathcal{F})) \cong R(F[\epsilon])$ .  $\square$

*Proof of Theorem 0.1 (i), (ii), (iv).* By Lemma 1.3, the condition  $(H_1)$  in [9, Theorem 2.11] holds. By Lemma 1.4 (b), the condition  $(H_2)$  holds. By Lemma 1.9, the tangent space of the functor  $R$  can be identified with  $H^1(X, j_*\mathcal{E}nd(\mathcal{F}))$ , which is finite dimensional over  $F$  by [4, 1.1]. So the condition  $(H_3)$  holds. Thus the functor  $R$  has a pro-representable hull. Suppose furthermore that  $\mathrm{End}_{F[\pi_1(X - S, \bar{\eta})]}(F^r)$  consists of scalar multiplications. Then the condition  $(H_4)$  holds by Lemma 1.4 (a). Thus the functor  $R$  is pro-representable.  $\square$

## 2 Obstruction to deformation

Keep the notation of §1.

**Lemma 2.1.** *Suppose  $S$  is nonempty. For any torsion discrete  $\pi_1(X - S, \bar{\eta})$ -module  $M$ , we have  $H^i(\pi_1(X - S, \bar{\eta}), M) = 0$  for all  $i \geq 2$ .*

*Proof.* Let  $I$  be a torsion discrete induced  $\pi_1(X - S, \bar{\eta})$ -module such that we have an embedding  $M \hookrightarrow I$ . We have  $H^i(\pi_1(X - S, \bar{\eta}), I) = 0$  for all  $i \geq 1$ . (Confer [10, I 2.5].) So

$$H^i(\pi_1(X - S, \bar{\eta}), M) \cong H^{i-1}(\pi_1(X - S, \bar{\eta}), I/M).$$

By induction on  $i$ , to prove the lemma, it suffices to show  $H^2(\pi_1(X - S, \bar{\eta}), M) = 0$  for any torsion discrete  $\pi_1(X - S, \bar{\eta})$ -module  $M$ . As  $H^2(\pi_1(X - S, \bar{\eta}), I) = 0$ , it suffices to prove that the map

$$H^1(\pi_1(X - S, \bar{\eta}), I) \rightarrow H^1(\pi_1(X - S, \bar{\eta}), I/M)$$

is surjective. Since  $I$  and  $M$  are torsion discrete  $\pi_1(X - S, \bar{\eta})$ -modules, they define  $F$ -sheaves  $\mathcal{I}$  and  $\mathcal{M}$  on  $X - S$ , respectively. By Lemma 1.6, we have

$$\begin{aligned} H^1(\pi_1(X - S, \bar{\eta}), I) &\cong H^1(X - S, \mathcal{I}), \\ H^1(\pi_1(X - S, \bar{\eta}), I/M) &\cong H^1(X - S, \mathcal{I}/\mathcal{M}). \end{aligned}$$

So it suffices to show that the map

$$H^1(X - S, \mathcal{I}) \rightarrow H^1(X - S, \mathcal{I}/\mathcal{M})$$

is surjective. We have an exact sequence

$$H^1(X - S, \mathcal{I}) \rightarrow H^1(X - S, \mathcal{I}/\mathcal{M}) \rightarrow H^2(X - S, \mathcal{M}).$$

Since  $S$  is nonempty,  $X - S$  is an affine curve. So we have  $H^2(X - S, \mathcal{M}) = 0$  by [1, XIV 3.2]. Our assertion follows.  $\square$

Suppose  $S$  is nonempty. Let  $A' \rightarrow A$  be an epimorphism in the category  $\mathcal{C}$  such that its kernel  $\mathfrak{a}$  has the property  $\mathfrak{m}_{A'}\mathfrak{a} = 0$ . Let  $\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A^r)$  be a representation such that  $\rho \equiv \rho_0 \pmod{\mathfrak{m}_A}$ . Fix a set theoretic continuous lifting  $\gamma : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A'^r)$  of  $\rho$ . Consider the map

$$\begin{aligned} c : \pi_1(X - S, \bar{\eta}) \times \pi_1(X - S, \bar{\eta}) &\rightarrow \mathfrak{a} \otimes_F \mathrm{End}(F^r) \cong \mathfrak{a} \otimes_F \mathrm{Ad}(\rho_0), \\ c(g_1, g_2) &= \gamma(g_1 g_2) \gamma(g_2)^{-1} \gamma(g_1)^{-1} - 1. \end{aligned}$$

One can show  $c$  is a 2-cocycle. By Lemma 2.1,  $c$  must be a 2-coboundary. Choose a continuous map

$$\delta : \pi_1(X - S, \bar{\eta}) \rightarrow \mathfrak{a} \otimes_k \mathrm{Ad}(\rho_0)$$

such that  $c = d(\delta\gamma^{-1})$ . Then  $\rho' = \gamma + \delta : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A'^r)$  is a representation lifting  $\rho$ . We have thus proved that  $\rho$  can always be lifted to a representation  $\rho' : \pi_1(X - S, \bar{\eta}) \rightarrow \mathrm{GL}(A'^r)$ .

Suppose furthermore that  $\rho|_{\text{Gal}(\bar{\eta}_s/\eta_s)} \cong \rho_0|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$  for any  $s \in S$ . Choose  $P_s \in \text{GL}(A^r)$  such that  $P_s^{-1}\rho(g)P_s = \rho_0(g)$  for all  $g \in \text{Gal}(\bar{\eta}_s/\eta_s)$ . Choose  $P'_s \in \text{GL}(A'^r)$  lifting  $P_s$ . Then  $(P'_s\rho_0 P'^{-1}_s)|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$  is a lifting of  $\rho|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$ . Now  $\rho'|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$  is also a lifting of  $\rho|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$ . As in the proof of Lemma 1.7, the continuous map  $\delta_s : \text{Gal}(\bar{\eta}_s/\eta_s) \rightarrow \text{Ad}(\rho_0) \otimes_F \mathfrak{a}$  defined by

$$\rho'(g) = P'_s\rho_0(g)P'^{-1}_s + \delta_s(g)P'_s\rho_0(g)P'^{-1}_s \quad (g \in \text{Gal}(\bar{\eta}_s/\eta_s))$$

is a 1-cocycle. Let  $[\delta_s]$  be the cohomology class of  $\delta_s$  in  $H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0) \otimes_F \mathfrak{a})$  and let  $c$  be the image of  $([\delta_s])_{s \in S}$  in the cokernel of the canonical homomorphism

$$H^1(\pi_1(X - S, \bar{\eta}), \text{Ad}(\rho_0) \otimes_F \mathfrak{a}) \rightarrow \bigoplus_{s \in S} H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0) \otimes_F \mathfrak{a}).$$

By Lemma 1.8 and Lemma 1.6, this cokernel can be considered as a subspace of  $H^2(X, j_*\mathcal{E}nd(\mathcal{F})) \otimes_F \mathfrak{a}$ . So we can also regard  $c$  as an element of in  $H^2(X, j_*\mathcal{E}nd(\mathcal{F})) \otimes_F \mathfrak{a}$ . We call  $c$  the *obstruction class to lifting  $\rho$  while preserving local data*. For simplicity, in the sequel we simply call  $c$  the obstruction class to lifting  $\rho$ . In Lemma 2.2 below, we will show that  $c$  is independent of the choice of  $\rho'$ ,  $P_s$  and  $P'_s$ . Note that we have

$$\begin{aligned} \det(\rho'(g)) &= \det\left((I + \delta_s(g))P'_s\rho_0(g)P'^{-1}_s\right) \\ &= (1 + \text{Tr}(\delta_s(g)))\det(\rho_0(g)) \\ &= \det(\rho_0(g)) + \text{Tr}(\delta_s(g))\det(\rho_0(g)). \end{aligned}$$

It follows that the obstruction class to lifting  $\det(\rho)$  is the image of the obstruction class to lifting  $\rho$  under the homomorphism

$$H^2(X, j_*\mathcal{E}nd(\mathcal{F})) \otimes_F \mathfrak{a} \rightarrow H^2(X, F) \otimes_F \mathfrak{a}$$

induced by

$$\text{Tr} : \mathcal{E}nd(\mathcal{F}) \rightarrow F.$$

**Lemma 2.2.** *Suppose  $S$  is nonempty. Let  $\phi : A' \rightarrow A$  be an epimorphism in the category  $\mathcal{C}$  such that its kernel  $\mathfrak{a}$  has the property  $\mathfrak{m}_{A'}\mathfrak{a} = 0$ . Let  $\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \text{GL}(A^r)$  be a representation such that  $\rho \equiv \rho_0 \pmod{\mathfrak{m}_A}$  and  $\rho|_{\text{Gal}(\bar{\eta}_s/\eta_s)} \cong \rho_0|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$  for any  $s \in S$ . Let  $\rho' : \pi_1(X - S, \bar{\eta}) \rightarrow \text{GL}(A'^r)$  be a representation lifting  $\rho$  (which always exists) and define the obstruction class  $c$  to lifting  $\rho$  as above.*

(i)  *$c$  is independent of the choice of  $\rho'$ ,  $P_s$  and  $P'_s$ , and  $c$  vanishes if and only if  $\rho$  can be lifted to a representation  $\rho'' : \pi_1(X - S, \bar{\eta}) \rightarrow \text{GL}(A'^r)$  such that  $\rho'' \equiv \rho_0 \pmod{\mathfrak{m}_A}$  and  $\rho''|_{\text{Gal}(\bar{\eta}_s/\eta_s)} \cong \rho_0|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$  for any  $s \in S$ .*

(ii) The obstruction class to lifting  $\det(\rho)$  is the image of  $c$  under the homomorphism

$$H^2(X, j_* \mathcal{E}nd(\mathcal{F})) \otimes_F \mathfrak{a} \rightarrow H^2(X, F) \otimes_F \mathfrak{a}$$

induced by  $\text{Tr} : \mathcal{E}nd(\mathcal{F}) \rightarrow F$ .

*Proof.* We have shown (ii) above. Let us prove (i). Let  $\rho'' : \pi_1(X - S, \bar{\eta}) \rightarrow \text{GL}(A^r)$  be another lifting of  $\rho$ , and define 1-cocycles

$$\delta_s, \theta_s : \text{Gal}(\bar{\eta}_s/\eta_s) \rightarrow \text{Ad}(\rho_0) \otimes_F \mathfrak{a}$$

by

$$\begin{aligned} \rho'(g) &= P'_s \rho_0(g) P_s'^{-1} + \delta_s(g) P'_s \rho_0(g) P_s'^{-1}, \\ \rho''(g) &= P'_s \rho_0(g) P_s'^{-1} + \theta_s(g) P'_s \rho_0(g) P_s'^{-1} \end{aligned}$$

for all  $g \in \text{Gal}(\bar{\eta}_s/\eta_s)$ . Since  $\rho'$  and  $\rho''$  are liftings of  $\rho$ , the continuous map

$$\psi : \pi_1(X - S, \bar{\eta}) \rightarrow \text{Ad}(\rho_0) \otimes_F \mathfrak{a}$$

defined by

$$\rho''(g) = \rho'(g) + \psi(g) \rho'(g) \quad (g \in \pi_1(X - S, \bar{\eta}))$$

is a 1-cocycle for the group  $\pi_1(X - S, \bar{\eta})$ . For any  $g \in \text{Gal}(\bar{\eta}_s/\eta_s)$ , we have

$$\begin{aligned} (\theta_s(g) - \delta_s(g)) P'_s \rho_0(g) P_s'^{-1} &= \rho''(g) - \rho'(g) \\ &= \psi(g) \rho'(g) \\ &= \psi(g) (P'_s \rho_0(g) P_s'^{-1} + \delta_s(g) P'_s \rho_0(g) P_s'^{-1}) \\ &= \psi(g) P'_s \rho_0(g) P_s'^{-1}, \end{aligned}$$

where the last equality follows from the fact that  $\mathfrak{a}^2 = 0$ . It follows that

$$\theta_s(g) - \delta_s(g) = \psi(g)$$

for all  $g \in \text{Gal}(\bar{\eta}_s/\eta_s)$ . Hence the cohomology class  $[\theta_s] - [\delta_s]$  is the image of the cohomology class  $[\psi]$  under the canonical homomorphism

$$H^1(\pi_1(X - S, \bar{\eta}), \text{Ad}(\rho_0) \otimes_F \mathfrak{a}) \rightarrow H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0) \otimes_F \mathfrak{a}).$$



It follows that  $([\theta_s])_{s \in S}$  and  $([\delta_s])_{s \in S}$  define the same element in the cokernel of the canonical homomorphism

$$H^1(\pi_1(X - S, \bar{\eta}), \text{Ad}(\rho_0) \otimes_F \mathfrak{a}) \rightarrow \bigoplus_{s \in S} H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0) \otimes_F \mathfrak{a}).$$

So the obstruction class to lifting  $\rho$  is independent of the choice of the lifting  $\rho'$  of  $\rho$ .

Choose  $\tilde{P}_s \in \text{GL}(A^r)$  such that  $\tilde{P}_s^{-1} \rho(g) \tilde{P}_s = \rho_0(g)$  for all  $g \in \text{Gal}(\bar{\eta}_s/\eta_s)$  and choose  $\tilde{P}'_s \in \text{GL}(A^r)$  lifting  $\tilde{P}_s$ . As representations of  $\text{Gal}(\bar{\eta}_s/\eta_s)$ ,  $\tilde{P}'_s \rho_0 \tilde{P}'_s^{-1}$  and  $P'_s \rho_0 P'^{-1}_s$  are equivalent and hence strictly equivalent relative to  $\phi$  by Lemma 1.5 (ii). Define

$$\delta''_s : \text{Gal}(\bar{\eta}_s/\eta_s) \rightarrow \text{Ad}(\rho_0) \otimes_F \mathfrak{a}$$

by

$$\tilde{P}'_s \rho_0(g) \tilde{P}'_s^{-1} = P'_s \rho_0(g) P'^{-1}_s + \delta''_s(g) P'_s \rho_0(g) P'^{-1}_s$$

for all  $g \in \text{Gal}(\bar{\eta}_s/\eta_s)$ . Then  $\delta''_s$  is a 1-coboundary. Define 1-cocycles

$$\delta_s, \tilde{\delta}_s : \text{Gal}(\bar{\eta}_s/\eta_s) \rightarrow \text{Ad}(\rho_0) \otimes_F \mathfrak{a}$$

by

$$\begin{aligned} \rho'(g) &= P'_s \rho_0(g) P'^{-1}_s + \delta_s(g) P'_s \rho_0(g) P'^{-1}_s, \\ \rho'(g) &= \tilde{P}'_s \rho_0(g) \tilde{P}'_s^{-1} + \tilde{\delta}_s(g) \tilde{P}'_s \rho_0(g) \tilde{P}'_s^{-1} \end{aligned}$$

for all  $g \in \text{Gal}(\bar{\eta}_s/\eta_s)$ . Then we have

$$\begin{aligned} \rho'(g) &= \tilde{P}'_s \rho_0(g) \tilde{P}'_s^{-1} + \tilde{\delta}_s(g) \tilde{P}'_s \rho_0(g) \tilde{P}'_s^{-1} \\ &= P'_s \rho_0(g) P'^{-1}_s + \delta''_s(g) P'_s \rho_0(g) P'^{-1}_s + \tilde{\delta}_s(g) (P'_s \rho_0(g) P'^{-1}_s + \delta''_s(g) P'_s \rho_0(g) P'^{-1}_s) \\ &= P'_s \rho_0(g) P'^{-1}_s + (\delta''_s(g) + \tilde{\delta}_s(g)) P'_s \rho_0(g) P'^{-1}_s. \end{aligned}$$

It follows that

$$\delta_s = \delta''_s + \tilde{\delta}_s$$

and hence  $\delta_s$  and  $\tilde{\delta}_s$  differ by a 1-coboundary. So the obstruction class to lifting  $\rho$  is independent of the choice of  $P_s$  and  $P'_s$ .

Suppose  $\rho$  can be lifted to a representation  $\rho'' : \pi_1(X - S, \bar{\eta}) \rightarrow \text{GL}(A^r)$  such that  $\rho'' \equiv \rho_0 \pmod{\mathfrak{m}_A}$  and  $\rho''|_{\text{Gal}(\bar{\eta}_s/\eta_s)} \cong \rho_0|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$  for any  $s \in S$ . Then  $\rho''|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$  is equivalent to  $P'_s \rho_0|_{\text{Gal}(\bar{\eta}_s/\eta_s)} P'^{-1}_s$ . By Lemma 1.5 (ii),  $\rho''|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$  and  $P'_s \rho_0|_{\text{Gal}(\bar{\eta}_s/\eta_s)} P'^{-1}_s$  are strictly equivalent relative to  $\phi'$ . By

Lemma 1.7, the 1-cocycle  $\theta_s$  defined above becomes a 1-coboundary for the group  $\text{Gal}(\bar{\eta}_s/\eta_s)$ . By the above discussion, to define the obstruction class to lifting  $\rho$ , we can use the lifting  $\rho''$  instead of the lifting  $\rho'$ . It follows that the obstruction class vanishes.

Conversely, suppose the obstruction class  $c$  to lifting  $\rho$  vanishes. Then we can find a 1-cocycle  $\psi : \pi_1(X - S, \bar{\eta}) \rightarrow \text{Ad}(\rho_0) \otimes_F \mathfrak{a}$  such that  $\psi|_{\text{Gal}(\bar{\eta}_s/\eta_s)} + \delta_s$  are 1-coboundaries for all  $s \in S$ . Set

$$\rho'' = \rho' + \psi\rho'.$$

Then  $\rho''$  is a lifting of  $\rho$ . Moreover, for any  $g \in \text{Gal}(\bar{\eta}_s/\eta_s)$ , we have

$$\begin{aligned} \rho''(g) &= \rho'(g) + \psi(g)\rho'(g) \\ &= P'_s\rho_0(g)P'^{-1}_s + \delta_s(g)P'_s\rho_0(g)P'^{-1}_s + \psi(g)(P'_s\rho_0(g)P'^{-1}_s + \delta_s(g)P'_s\rho_0(g)P'^{-1}_s) \\ &= P'_s\rho_0(g)P'^{-1}_s + (\psi(g) + \delta_s(g))P'_s\rho_0(g)P'^{-1}_s. \end{aligned}$$

Since  $\psi|_{\text{Gal}(\bar{\eta}_s/\eta_s)} + \delta_s$  is a 1-coboundary for each  $s \in S$ ,  $\rho''|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$  must be strictly equivalent to  $P'_s\rho_0(g)P'^{-1}_s$  relative to  $\phi'$  by Lemma 1.7. In particular,  $\rho''|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$  is equivalent to  $\rho_0|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$ .  $\square$

**Lemma 2.3.** *Suppose  $S$  is nonempty. Let  $\phi : A' \rightarrow A$  be an epimorphism in the category  $\mathcal{C}$  such that its kernel  $\mathfrak{a}$  has the property  $\mathfrak{m}_{A'}\mathfrak{a} = 0$ , and let  $\rho : \pi_1(X - S, \bar{\eta}) \rightarrow \text{GL}(A^r)$  be a representation such that  $\rho \equiv \rho_0 \pmod{\mathfrak{m}_A}$  and  $\rho|_{\text{Gal}(\bar{\eta}_s/\eta_s)} \cong \rho_0|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$  for any  $s \in S$ . If the rank  $r$  of  $\rho_0$  is 1, then the obstruction class to lifting  $\rho$  vanishes.*

*Proof.* By Lemma 2.2 (i), it suffices to show that  $R(A') \rightarrow R(A)$  is surjective. For any  $A \in \text{ob } \mathcal{C}$ , let  $R'(A)$  be the set of representations  $\rho : \pi_1(X, \bar{\eta}) \rightarrow A^*$  such that  $\rho \equiv 1 \pmod{\mathfrak{m}_A}$ . (Here we work with representations of  $\pi_1(X, \bar{\eta})$ , not those of  $\pi_1(X - S, \bar{\eta})$ . Recall that two rank 1 representations are equivalent if and only if they are equal.) Note that  $R'(A)$  can be identified with the set of rank 1 representations  $\rho$  of  $\pi_1(X - S, \bar{\eta})$  such that  $\rho \equiv 1 \pmod{\mathfrak{m}_A}$  and  $\rho|_{\text{Gal}(\bar{\eta}_s/\eta_s)} = 1$  for all  $s \in S$ .  $R'$  is a functor from  $\mathcal{C}$  to the category of sets, and we have an isomorphism of functors  $R' \xrightarrow{\cong} R$  defined by the map

$$R'(A) \xrightarrow{\cong} R(A), \quad \rho \mapsto \rho\rho_0$$

for each  $A \in \text{ob } \mathcal{C}$ . Let us prove  $R'(A') \rightarrow R'(A)$  is surjective for any epimorphism  $A' \rightarrow A$  in  $\mathcal{C}$ , that is, the functor  $R'$  is smooth. It suffices to prove the universal deformation ring  $R'_{\text{univ}}$  for the trivial representation  $1 : \pi_1(X, \bar{\eta}) \rightarrow F^*$  is a formal power series ring. Let  $A$  be an object in  $\mathcal{C}$ , and let

$\rho : \pi_1(X, \bar{\eta}) \rightarrow A^*$  be a representation such that  $\rho \equiv 1 \pmod{\mathfrak{m}_A}$ . Then the image of  $\rho$  is contained in the subgroup  $1 + \mathfrak{m}_A$  of  $A^*$ . This subgroup has a filtration

$$1 + \mathfrak{m}_A \supset 1 + \mathfrak{m}_A^2 \supset \cdots$$

For each  $i$ , we have an isomorphism of groups

$$\mathfrak{m}_A^i / \mathfrak{m}_A^{i+1} \cong (1 + \mathfrak{m}_A^i) / (1 + \mathfrak{m}_A^{i+1}),$$

and  $\mathfrak{m}_A^i / \mathfrak{m}_A^{i+1}$  is the underlying abelian group of a finite dimensional vector space over  $F$ . In the case where  $F$  is a finite extension of  $\mathbb{F}_\ell$  or  $\mathbb{Q}_\ell$ , any profinite subgroup of a finite dimensional  $F$ -vector space must be a pro- $\ell$ -group. It follows that the representation  $\rho : \pi_1(X, \bar{\eta}) \rightarrow A^*$  must factor through the pro- $\ell$ -completion of  $\pi_1(X, \bar{\eta})$  in this case. In the case where  $F = \overline{\mathbb{F}_\ell}$  or  $\overline{\mathbb{Q}_\ell}$ , we can find a finite extension  $E$  of  $\mathbb{F}_\ell$  or  $\mathbb{Q}_\ell$  and a local Artinian  $E$ -algebra  $A_E$  with residue field  $E$  such that  $A_E \otimes_E F \cong A$  and such that  $\rho : \pi_1(X, \bar{\eta}) \rightarrow A^*$  factors through  $\pi_1(X, \bar{\eta}) \rightarrow A_E^*$ . The above discussion shows that  $\rho : \pi_1(X, \bar{\eta}) \rightarrow A^*$  again factors through the pro- $\ell$ -completion of  $\pi_1(X, \bar{\eta})$ . As the representation is of rank 1,  $\rho$  factors through the abelianization  $\Gamma$  of the pro- $\ell$ -completion of  $\pi_1(X, \bar{\eta})$ . By [6, X 3.10], the pro- $\ell$ -completion of  $\pi_1(X, \bar{\eta})$  is isomorphic to the pro- $\ell$ -completion of the group with generators  $s_i, t_i$  ( $1 \leq i \leq g = \text{genus}(X)$ ) and with one relation

$$(s_1 t_1 s_1^{-1} t_1^{-1}) \cdots (s_g t_g s_g^{-1} t_g^{-1}) = 1.$$

It follows that  $\Gamma \cong \mathbb{Z}_\ell^{2g}$ . For any nonnegative integer  $m$ ,  $F[[t_1, \dots, t_{2g}]] / (t_1, \dots, t_{2g})^m$  is a finite dimensional vector space over  $F$ , and hence has a natural topology inherited from the topology on  $F$ . Endow  $F[[t_1, \dots, t_{2g}]] = \varprojlim_m F[[t_1, \dots, t_{2g}]] / (t_1, \dots, t_{2g})^m$  with the projective limit topology. Then the homomorphism

$$\gamma : \mathbb{Z}^{2g} \rightarrow (F[[t_1, \dots, t_{2g}]])^*, \quad (\lambda_1, \dots, \lambda_{2g}) \rightarrow (1 + t_1)^{\lambda_1} \cdots (1 + t_{2g})^{\lambda_{2g}}$$

is continuous if we put the  $\ell$ -adic topology on  $\mathbb{Z}^{2g}$ . So it induces a continuous homomorphism

$$\mathbb{Z}_\ell^{2g} \rightarrow (F[[t_1, \dots, t_{2g}]])^*.$$

One can then verify the ring  $F[[t_1, \dots, t_{2g}]]$  together with the representation  $\pi_1(X, \bar{\eta}) \rightarrow (F[[t_1, \dots, t_{2g}]])^*$  defined by the composite

$$\pi_1(X, \bar{\eta}) \rightarrow \Gamma \cong \mathbb{Z}_\ell^{2g} \xrightarrow{\gamma} (F[[t_1, \dots, t_{2g}]])^*$$

satisfies the universal property required for the universal deformation ring and the universal deformation. So the universal deformation ring is isomorphic to a formal power series ring. This finishes the proof of the lemma.  $\square$

*Proof of Theorem 0.1 (iii).* Note that the pairing

$$\mathcal{E}nd(\mathcal{F}) \times \mathcal{E}nd(\mathcal{F}) \rightarrow F, \quad (\phi, \psi) \mapsto \text{Tr}(\psi \circ \phi)$$

defines a self-duality on  $\mathcal{E}nd(\mathcal{F})$ . By the duality theorem ([3, 1.3 and 2.2]), we have a perfect pairing

$$H^2(X, j_* \mathcal{E}nd(\mathcal{F})) \times H^0(X, j_* \mathcal{E}nd(\mathcal{F})(1)) \rightarrow F.$$

If all elements in  $\text{End}_{F[\pi_1(X-S, \bar{\eta})]}(F^r)$  are scalar multiplications, then we have

$$F \cong \text{End}(\mathcal{F}) \cong H^0(X, j_* \mathcal{E}nd(\mathcal{F})).$$

So the morphism

$$F \rightarrow \mathcal{E}nd(\mathcal{F}), \quad a \mapsto aI$$

induces an isomorphism

$$H^0(X, F) \cong H^0(X, j_* \mathcal{E}nd(\mathcal{F})).$$

This implies that  $\text{Tr} : \mathcal{E}nd(\mathcal{F}) \rightarrow F$  induces an isomorphism

$$H^2(X, j_* \mathcal{E}nd(\mathcal{F})) \xrightarrow{\cong} H^2(X, F).$$

First consider the case where  $S$  is nonempty. By Lemma 2.2, this last isomorphism maps the obstruction class to lifting a deformation of  $\rho_0$  to the obstruction class to lifting a corresponding deformation of  $\det(\rho_0)$ . By Lemma 2.3, there is no obstruction to lifting a deformation of  $\det(\rho_0)$ . It follows that there is no obstruction to lifting a deformation of  $\rho_0$ . Hence the functor  $R$  is smooth, and the universal deformation ring  $R_{\text{univ}}$  is isomorphic to the formal power series ring  $F[[t_1, \dots, t_m]]$  with  $m = \dim_F H^1(X, j_* \mathcal{E}nd(\mathcal{F}))$ .

Next we consider the case where  $S$  is empty. Fix a closed point  $\infty$  in  $X$ . For any  $A \in \text{ob } \mathcal{C}$ , let  $R'(A)$  be the set of strict equivalent classes of representations  $\rho : \pi_1(X - \{\infty\}, \bar{\eta}) \rightarrow \text{GL}(A^r)$  such that  $\rho \equiv \rho_0 \pmod{\mathfrak{m}_A}$  and  $\rho|_{\text{Gal}(\bar{\eta}_\infty/\eta_\infty)} \cong \rho_0|_{\text{Gal}(\bar{\eta}_\infty/\eta_\infty)}$ . Note that  $\rho_0|_{\text{Gal}(\bar{\eta}_\infty/\eta_\infty)}$  is the trivial representation of rank  $r$ , and the functor  $R$  is isomorphic to  $R'$ . We are thus reduced to the case where  $S = \{\infty\}$  is nonempty.  $\square$

## References

- [1] M. Artin, A. Grothendieck, J. L. Verdier, *Théorie des Topos et Cohomologie Étale des Schémas* (SGA 4), Lecture Notes in Math. 269, 270, 305, Springer-Verlag (1972-1973).
- [2] S. Bloch and H. Esnault, Local Fourier transform and rigidity for  $\mathcal{D}$ -modules, *Asian J. Math.* 8 (2004), 587-606.
- [3] P. Deligne, Dualité, in *Cohomologie Étale* (SGA 4 $\frac{1}{2}$ ), Lecture Notes in Math. 569, Springer-Verlag (1977), 154-167.
- [4] P. Deligne, Théorèmes de finitude en cohomologie  $\ell$ -adique, in *Cohomologie Étale* (SGA 4 $\frac{1}{2}$ ), Lecture Notes in Math. 569, Springer-Verlag (1977), 233-261.
- [5] A. Grothendieck and J. Dieudonné, Étude globale élémentaire de quelques classes de morphismes (EGA III), *Publ. Math. IHES* 8 (1961).
- [6] A. Grothendieck, *Revêtements Étales et Groupe Fondamental* (SGA 1), Lecture Notes in Math. 224, Springer-Verlag (1971).
- [7] N. Katz, *Rigid Local Systems*, Annals of Math. Studies 139, Princeton University Press (1996).
- [8] B. Mazur, Deforming Galois representations, *Galois groups over  $\mathbb{Q}$* , 385-437, Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989.
- [9] M. Schlessinger, Functors on Artin rings, *Trans. A.M.S.* 130 (1968), 208-222.
- [10] J.-P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Math. 5, Springer-Verlag (1964).